Derivation of the Lorentz transformation without use of light

1968 J. Phys. A: Gen. Phys. 1 331

(http://iopscience.iop.org/0022-3689/1/3/304)

Derivation of the Lorentz transformation without use of light

H. ALMSTRÖM
Department of Theoretical Physics, Royal Institute of Technology, Stockholm, Sweden

MS. received 24*th November* 1967, *in revised form* 18*th January* 1968

Abstract. The connection between the length of a rigid rod in motion, L_m, and a rigid rod at rest, L_n, is assumed to be of the form $L_n = \alpha(v_{nm})L_m$ where v_{nm} is the relative velocity measured from the system where L_n is at rest. We shall show that (i) for small v_{nm}, we have $v_{nm} = -v_{mn}$, (ii) if this relation is assumed to hold for finite v_{nm}, we obtain the velocity-addition law and (iii) if $\alpha = \alpha|v_{nm}|$, the form of α is restricted to $\alpha = (1-Dv_{nm}^2)^{1/2}$, where D is a constant. When $D > 0$ we have that $v_g = (1/D)^{1/2}$ is a limit velocity.

1. Introduction

In this paper we shall analyse the assumptions made in the derivation of the Lorentz transformation. For this purpose it seems advisable to leave light out of the derivation.

We shall analyse the space–time transformation between inertial systems moving along parallel Euclidian straight lines starting from the assumptions: (i) the Galilean relativity principle (see for example Fock 1963), (ii) the length of a rod moving along one of the parallel lines and measured from one of the inertial systems shall depend on its velocity. The analysis is similar to that of Frank and Rothe (1911) who also obtain a generalized 'Lorentz factor'. The 'Lorentz factor' which we derive is obtained from that of Frank and Rothe by putting $\alpha_{11} = \alpha_{22} = b/c$ and $\alpha_{12} = -1/c^2$, a case which they do not discuss.

In a subsequent paper we shall analyse the possibility of deriving a theory with this generalized 'Lorentz factor', where we have the invariant

$$I^2 = \{(x^0)^2 - \mathbf{x}\,.\,\mathbf{x}\}\left|\frac{1-\hat{\mathbf{n}}\,.\,\mathbf{v}}{1+\hat{\mathbf{n}}\,.\,\mathbf{v}}\right|^b .$$

In the present analysis $\hat{\mathbf{n}}$ is the direction of the parallel lines. In the wave equation constructed from the invariant we try to replace $\hat{\mathbf{n}}$ by the Pauli spin matrices $\boldsymbol{\sigma}$.

Definition of relative velocity. Let v_{nm} be the relative velocity between two inertial systems K_n and K_m measured from K_n.

2. Length-change measurements

Two rigid rods with the lengths L_n and L_m are at rest in an inertial system K_n. The rod with length L_m is given the relative velocity v_{nm}, (i) in the direction of L_n, (ii) perpendicular to the direction of L_n, and its length is compared in K_n with the rod at rest there. The length L_m is adjusted so that the moving rod has the length L_n measured from K_n. The results of the measurements are assumed to be given by the relations $L_n{}'' = \alpha(v_{nm})L_m{}''$, where $\alpha(0) = 1$, for the parallel motion and $L_n{}^{\perp} = h(v_{nm})L_m{}^{\perp}$, where $h(0) = 1$, for the perpendicular motion.

3. Introduction of time

If we study the inertial systems K_n and K_m moving with constant relative velocity, whose origins coincide for $t_n = 0$, we obtain

$$x_n - v_{nm}t_n = \alpha(v_{nm})x_m \tag{1}$$

where the x axes are along parallel lines and point in the same direction.

Determination of the form for $\alpha(v_{nm})$. If we study further the inertial systems K_n, K_m and K_p moving with constant relative velocities and whose origins coincide for $t_n = t_m = t_p = 0$, we obtain from (1)

$$x_n - v_{nm} t_n = (v_{nm}) x_m \tag{2}$$

$$x_n - v_{np} t_n = \alpha(v_{np}) x_p \tag{3}$$

$$x_m - v_{mp} t_m = \alpha(v_{mp}) x_p \tag{4}$$

$$x_m - v_{mn} t_m = \alpha(v_{mn}) x_n \tag{5}$$

$$x_p - v_{pn} t_p = \alpha(v_{pn}) x_n \tag{6}$$

$$x_p - v_{pm} t_p = \alpha(v_{pm}) x_m . \tag{7}$$

From (2), (3) and (4) we obtain

$$\left\{\frac{\alpha(v_{np})\alpha(v_{mn})}{\alpha(v_{mp})} v_{nm} - \alpha(v_{nm})\alpha(v_{mn}) v_{np}\right\} \frac{1}{v_{nm} - v_{np}} x_m - \left\{\frac{\alpha(v_{np})\alpha(v_{mn})}{\alpha(v_{mp})} \frac{v_{mp} v_{nm}}{v_{nm} - v_{np}}\right\} t_m = \alpha(v_{mn}) x_n . \tag{8}$$

From (5) and (8) we obtain

$$\frac{\alpha(v_{mn})\alpha(v_{np})}{\alpha(v_{mp})} = \frac{v_{mn}(v_{nm} - v_{np})}{v_{mp} v_{nm}} \tag{9}$$

$$\alpha(v_{nm})\alpha(v_{mn}) - 1 = \frac{v_{nm} v_{mn} - v_{nm} v_{mp} - v_{np} v_{mn}}{v_{np} v_{mp}} . \tag{10}$$

Equation (10) can be rewritten as

$$\alpha(v_{mn}) \left\{\frac{\alpha(v_{nm}) - \alpha(0)}{v_{nm}}\right\} + \alpha(0) \left\{\frac{\alpha(v_{mn}) - \alpha(0)}{v_{nm}}\right\} - \frac{v_{mn}}{v_{np} v_{mp}} + \frac{v_{mp} - v_{np}}{v_{np} v_{mp}} = -\left(\frac{1}{v_{mp}}\right)\left(1 + \frac{v_{mn}}{v_{nm}}\right). \tag{11}$$

When $v_{nm} \to 0$ we assume that $v_{mn} \to 0$ and $v_{mp} - v_{np} \to 0$. The left-hand side of (11) will approach a v_{mp}-independent expression for decreasing v_{nm}; the right-hand side also approaches a v_{mp}-independent expression so long as $v_{mn} = -v_{nm}$ for small v_{nm}. From (11) we further see that $d\alpha(v_{nm})/dv_{nm}$ is a continuous function at $v_{nm} = 0$. If we now assume the relation

$$v_{nm} = -v_{mn} \tag{12}$$

to hold for all finite relative velocities, equation (10) may be rewritten as

$$\frac{1 - \alpha(v_{nm})\alpha(-v_{nm})}{v_{nm}^2} = \frac{v_{nm} + v_{mp} - v_{np}}{v_{nm} v_{mp} v_{np}} \tag{13}$$

From equations (2), (3), (6) and (7) we obtain in the same way

$$\frac{1 - \alpha(v_{np})\alpha(-v_{np})}{v_{np}^2} = \frac{v_{np} + v_{pm} - v_{nm}}{v_{np} v_{pm} v_{nm}} . \tag{14}$$

Using (12), (13) and (14) we obtain

$$\frac{1 - \alpha(v_{nm})\alpha(-v_{nm})}{v_{nm}^2} = \frac{1 - \alpha(v_{np})\alpha(-v_{np})}{v_{np}^2} = D \tag{15}$$

where D must be a constant. Equations (14) and (15) now give the velocity-addition law

$$v_{np} = \frac{v_{nm} + v_{mp}}{1 + D v_{nm} v_{mp}} . \tag{16}$$

For $D > 0$ we obtain from (16) that $v_g = (1/D)^{1/2}$ is a limit velocity which we identify with the velocity of light c *in vacuo.*

If we introduce the notation $v/v_g = a$, we obtain from (9), (15) and (16)

$$\frac{\alpha(a_{nm})\alpha(a_{mp})}{\alpha(a_{np})(1+a_{nm}a_{mp})} = 1. \tag{17}$$

If we put $\alpha(a_{nm}) = \exp\{g(a_{nm})\}(1-a_{nm}{}^2)^{1/2}$, we obtain from (16) and (17)

$$g\left(\frac{a_{nm}+a_{mp}}{1+a_{nm}a_{mp}}\right) - g(a_{nm}) = g(a_{mp}). \tag{18}$$

Let us divide (18) by a_{mp} and let $a_{mp} \to 0$. This gives

$$(1-a_{nm}{}^2)\frac{dg(a_{nm})}{da_{nm}} = \lim_{a_{mp}\to 0} \frac{g(a_{mp})}{a_{mp}} = b \tag{19}$$

where b must be a constant. We have $\alpha(0) = 1$ which gives $g(0) = 0$. The solution to (19) is then

$$\exp\{g(a_{nm})\} = \left(\frac{1+a_{nm}}{1-a_{nm}}\right)^{b/2}. \tag{20}$$

For $\alpha(v_{nm})$ we now obtain

$$\alpha(v_{nm}) = \left|\frac{1+v_{nm}/c}{1-v_{nm}/c}\right|^{b/2} \left\{1-\left(\frac{v_{nm}}{c}\right)^2\right\}^{1/2} \tag{21}$$

as from (16) $|v_{nm}| \leqslant c$.

If we now assume that there is no polarization of the space, we have

$$\alpha = \alpha(|v_{nm}|). \tag{22}$$

Equation (22) gives $b = 0$, and we obtain

$$\alpha(|v_{nm}|) = \left\{1-\left(\frac{v_{nm}}{c}\right)^2\right\}^{1/2}. \tag{23}$$

From (2), (5) and (23) we obtain the Lorentz transformation

$$x_m = \left\{1-\left(\frac{v_{nm}}{c}\right)^2\right\}^{-1/2} (x_n - v_{nm}t_n)$$

$$t_m = \left\{1-\left(\frac{v_{nm}}{c}\right)^2\right\}^{-1/2} \left(t_n - \frac{v_{nm}x_n}{c^2}\right).$$

References

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